J. Math. Anal. Appl. 348 (2008) 996–1011

Contents lists available at ScienceDirect

Journal of Mathematical Analysis and Applications

www.elsevier.com/locate/jmaa

# Estimates for Emden–Fowler type inequalities with absorption term

Mikhail Surnachev

*Department of Mathematics, Swansea University, Swansea SA2 8PP, UK*

A R T I C L E   I N F O

*Article history:*
Received 12 March 2008
Available online 13 August 2008
Submitted by P.J. McKenna

*Keywords:*
Keller–Osserman type estimates
Semilinear elliptic equations
Maximum principle
Sub- and supersolutions
Singularities
Exterior domains
Nonlinear absorption

A B S T R A C T

We obtain improved estimates of the Keller–Osserman type for second-order elliptic semilinear inequalities in the non-divergent form $\operatorname{sign}(u)(a_{ij}(x)u_{x_ix_j} + b_i(x)u_{x_i}) \geqslant c(x)|x|^{-2}|u|^\sigma$. Special cases of rapidly growing or decaying weights $c(x)$ and planar domains are also treated.



## 1. Introduction, main results and applications

In this paper we study the behaviour of solutions to the Emden–Fowler type inequality with absorbtion term:

$$\operatorname{sign}(u)\mathcal{L}u \geqslant \frac{c(x)}{|x|^2}|u|^\sigma, \quad x \in \Omega \subset \mathbb{R}^n, \tag{1.1}$$

where $\Omega$ is either the exterior of a ball or a ball punctured at the origin. In (1.1) $\sigma > 1$ is a constant and $\mathcal{L}$ is the elliptic operator in non-divergence form

$$\mathcal{L}u = \sum_{i,j=1}^n a_{ij}(x)u_{x_ix_j} + \sum_{i=1}^n b_i(x)u_{x_i}. \tag{1.2}$$

The coefficients $a_{ij}, b_i, c : \Omega \to \mathbb{R}$ are locally bounded measurable functions satisfying the conditions

(i) there exist $\nu_1, \nu_2 > 0$ such that

$$\nu_1|\xi|^2 \leqslant \sum_{i,j=1}^n a_{ij}(x)\xi_i\xi_j \leqslant \nu_2|\xi|^2 \quad \text{for all } x \in \Omega,\ \xi \in \mathbb{R}^n, \tag{1.3}$$

(ii)
$$\sup_{x\in\Omega}\left|\sum_{i=1}^n b_i(x)x_i\right| < +\infty, \quad \text{and} \tag{1.4}$$

(iii)
$$\inf_{x\in K} c(x) > 0 \quad \text{for any compact subset } K \Subset \Omega. \tag{1.5}$$

*E-mail address:* mamsu@swansea.ac.uk.

0022-247X/$ – see front matter 
doi:10.1016/j.jmaa.2008.08.013

We say that $u \in W^{2,n}_{\mathrm{loc}}(\Omega)$ is a solution to (1.1) in $\Omega$ if $u$ satisfies Eq. (1.1) pointwise almost everywhere (a.e.) in $\Omega$. Similarly, $u \in W^{2,n}_{\mathrm{loc}}(\Omega)$ is a supersolution to (1.1) in $\Omega$ if $u$ satisfies

$$\operatorname{sign}(u)\mathcal{L}u \leqslant \frac{c(x)}{|x|^2}|u|^\sigma \quad \text{a.e. in } \Omega.$$

By $B_r(x)$ we denote the ball of radius $r$ with the centre at point $x$, i.e. $B_r(x) = \{y \in \mathbb{R}^n\colon\ |y-x| < r\}$, $B_r := B_r(0)$. $A_{\rho_1,\rho_2}$ stands for an open annulus $A_{\rho_1,\rho_2} = \{x \in \mathbb{R}^n\colon\ \rho_1 < |x| < \rho_2\}$, where $\rho_1, \rho_2 \in [0, +\infty]$. Thus, $A_{\rho,\infty} = \mathbb{R}^n/\overline{B_\rho}$ and $A_{0,\rho} = B_\rho/\{0\}$ .

We introduce the following functions which are used to state the properties of $\mathcal{L}$:

$$T(x) = \sum_{i=1}^n a_{ii}(x) + \sum_{i=1}^n b_i(x)x_i, \qquad \Phi(x) = \sum_{i,j=1}^n a_{ij}(x)\frac{x_i x_j}{|x|^2}, \qquad A(x) = \frac{T(x)}{\Phi(x)}.$$

The last function was introduced in [16] where it was called the "effective dimension." It turns out that it plays an important role not only in describing properties of the corresponding linear equation but also in studying nonlinear equation of Emden–Fowler type [8,9]. The following notation is standard

$$u_+ = \max(u, 0), \qquad u_- = \max(-u, 0).$$

We also denote $M(r) = \sup_{|x|=r} u(x)$.

Inequalities of type (1.1) are of great importance in many areas of mathematical physics and for a long time have been attracting attention of many authors. The qualitative theory of this type of equations has a rich mathematical structure and yields a lot of beautiful results. One of the interesting and popular questions in this theory is a study of singularities of solutions to equations and inequalities of type (1.1) and their behavior in exterior domains. The tool whose value is hard to overestimate is widely known as the Keller–Osserman estimate. For the equation of the form

$$\Delta u = u^\sigma \quad \text{in } \Omega \tag{1.6}$$

it was first established in the works of the named authors [5,17] and reads as follows. Suppose $u \in C^2(\Omega)$ is a solution to (1.6). Then there exists a constant $C = C(\sigma, n)$ such that

$$\bigl|u(x)\bigr| \leqslant C\bigl(\operatorname{dist}(x, \partial\Omega)\bigr)^{\frac{2}{1-\sigma}}. \tag{1.7}$$

If $u$ is a solution to (1.6) in $A_{\rho,\infty}$, the last inequality immediately implies that $u(x) \to 0$ as $x \to \infty$, and

$$\bigl|u(x)\bigr| \leqslant C|x|^{\frac{2}{1-\sigma}}, \quad x \in A_{2\rho,\infty}.$$

Property (1.7) of solutions of (1.6) is a feature inherent to this class of nonlinear equations and with its help many results concerning behaviour of solutions of (1.6) are derived and obtaining existence results is significantly simplified. Moreover, in some cases this a priori bound leads to the removability of isolated singularities [2,7] or to their fairly complete description (see [10,21–23] and references therein). This estimate was generalised to divergent and non-divergent elliptic operators of the form $\frac{\partial}{\partial x_i}(a_{ij}\frac{\partial}{\partial x_j})$, $a_{ij}\frac{\partial^2}{\partial x_i \partial x_j}$ respectively in [7] and for parabolic equations and inequalities in [3]. The latter work studies even more general case of the nonlinearity $f(u)$ with the function $f$ satisfying certain structure conditions. In both [7] and [3] the differential operators do not contain any lower order terms or weight standing by nonlinearity. To some extent, via a use of the scaling method, the weight in front of the nonlinearity can be tackled. We show now the point at which the problems arise.

For simplicity, let $u$ be a positive solution to the equation

$$\Delta u = \frac{c(x)}{|x|^2}u^\sigma \tag{1.8}$$

in $A_{\rho,\infty}$. For $x \in A_{\delta\rho,\infty}$, $\delta > 1$, consider the ball $B := B_{|x|/\delta}(x)$. Substituting $u = \beta v$ we see that in this ball

$$\Delta v \geqslant \beta^{\sigma-1}\left(\inf_{y\in B}\frac{c(y)}{|y|^2}\right)v^\sigma = v^\sigma,$$

if we choose $\beta = (\inf_{y\in B}\frac{c(y)}{|y|^2})^{\frac{1}{1-\sigma}}$. Now estimate (1.7) yields $v(x) \leqslant C|x|^{\frac{2}{1-\sigma}}$, and hence

$$\bigl|u(x)\bigr| \leqslant C\Bigl(\min_{|y-x|<|x|/\delta} c(y)\Bigr)^{\frac{1}{1-\sigma}}. \tag{1.9}$$

We will refer further to this argument as the "scaling argument." If we take $c(x) \equiv 1$, in this way we obtain only boundedness and if $c(x) \to 0$ as $x \to \infty$ estimate (1.9) may fail to produce the right answer as is shown in our examples below. The cases when it happens present often the special interest as "critical cases."

Great work in this direction, and in much greater generality than is present here, was done by A. Kon'kov (see monograph [12], also [11,14,15]). His proofs rely on subtle integral estimates and comparison theorems which reduce studying *positive* solutions to (1.1) to studying positive solutions to the corresponding ODE of the same type. This approach was carried over to the study of nonlinear parabolic equations and inequalities.

The aim of this paper is to cover the remaining gaps and to obtain the sharp estimates in cases which cannot be covered by the scaling argument. Moreover, we present here a proof which is elementary in nature and relies entirely on the maximum principle [4,18] and explicit construction of supersolutions. The idea in [7] (and in [3]) was to write a supersolution in the form $\sum_{i=1}^n v(x_i)$, where $v$ was defined as a solution to the appropriate differential equation. In case when the differential operator contains no lower order terms the function $v$ is obtained as a solution to the differential equation

$$v'' = \lambda v^\sigma,$$

which can be easily integrated. Of course, when the lower order terms and the weight in front of the nonlinearity are present it becomes more complex and needs a great amount of subtle analysis to deal with. Here we show how to avoid this difficulties by taking only the leading terms of the asymptotic expansion of the corresponding differential equation [1,6,13,19]. It is very likely that the same method can be extended to parabolic equations and to equations with nonlinearity in the principal part ([20], [15], etc.)—the first point of interest in this case is obtaining the a priori bound for solutions of $u_t = \Delta u - |x|^{-2}u^\sigma$. The author plans to continue research in this direction.

Now we are ready to formulate the main results of the paper. In the statements of the theorems of this paper $C$ stands for a constant independent of $u$, whose value varies from line to line.

**Theorem 1.1.** *Let $u$ be a solution to inequality* (1.1) *in $A_{\rho,\infty}$. Let $Q(\cdot):(\rho,+\infty)\to\mathbb{R}^+$ be such that*

$$Q(r) \leqslant \inf_{|x|=r} c(x), \qquad \sup_{r>\rho}\left|\frac{Q'r}{Q}\right| < +\infty.$$

(a) *If $\int_\rho^{+\infty} Q(r)\frac{dr}{r} = +\infty$, then*

$$\left|u(x)\right| \leqslant C\left(\int_\rho^{|x|} Q(r)\frac{dr}{r}\right)^{\frac{1}{1-\sigma}}, \quad x\in A_{2\rho,\infty}, \tag{1.10}$$

*and as a consequence*

$$u(x)\to 0 \quad \text{as } x\to\infty.$$

(b) *If $\int_\rho^{+\infty} Q(r)\frac{dr}{r} < +\infty$ and $Q$ is bounded, then*

$$\left|u(x)\right| \leqslant C\left(\int_{|x|}^{+\infty} Q(r)\frac{dr}{r}\right)^{\frac{1}{1-\sigma}}, \quad x\in A_{2\rho,\infty}. \tag{1.11}$$

The next theorem is a generalisation of the previous one to the case of non-smooth weight.

**Theorem 1.2.** *Let $u$ be a solution to inequality* (1.1) *in $A_{\rho,\infty}$. Let $q(\cdot):(\rho,+\infty)\to\mathbb{R}^+$ be such that*

$$q(r) \leqslant \inf_{x\in\bar{A}_{r/\delta,r\delta}} c(x) \quad \text{for some } \delta > 1.$$

(a) *If $\int_\rho^{+\infty} q(r)\frac{dr}{r} = +\infty$ and*

$$\sup_{r>\rho}\frac{q(r\delta)}{q(r)} < +\infty,$$

*then*

$$\left|u(x)\right| \leqslant C\left(\int_\rho^{|x|} Q(r)\frac{dr}{r}\right)^{\frac{1}{1-\sigma}}, \quad x\in A_{2\rho,\infty}. \tag{1.12}$$

(b) *If $\int_\rho^{+\infty} q(r)\frac{dr}{r} < +\infty$, $q$ is bounded and*

$$\sup_{r>\rho}\frac{q(r)}{q(r\delta)} < +\infty,$$

*then*

$$\left|u(x)\right| \leqslant C\left(\int_{|x|}^{+\infty} q(r)\frac{dr}{r}\right)^{\frac{1}{1-\sigma}}, \quad x \in A_{2\rho,\infty}. \tag{1.13}$$

The third theorem exposes the effect emerging when the weight $c(x)$ is rapidly growing or decaying. (roughly speaking, faster than any power of $|x|$).

**Theorem 1.3.** *Let $u$ be a solution to inequality* (1.1) *in $A_{\rho,\infty}$. Let $Q(\cdot):(\rho,+\infty)\to\mathbb{R}^+$ be such that*

$$Q(r) \leqslant \inf_{|x|=r} c(x).$$

(a) *Suppose that*

$$\frac{rQ'}{Q} \to +\infty \quad \text{as } r \to +\infty, \quad \text{and}$$

$$Q', Q'' \geqslant 0, \quad \sup_{r>2\rho} \frac{QQ''}{(Q')^2} < +\infty.$$

*Then*

$$u(x) \to 0 \quad \text{as } x \to \infty,$$

*and*

$$\left|u(x)\right| \leqslant C\left(\int_{\rho}^{|x|} \frac{Q^2}{rQ'}\frac{dr}{r}\right)^{\frac{1}{1-\sigma}}, \quad x \in A_{2\rho,\infty}. \tag{1.14}$$

(b) *Suppose that*

$$\frac{rQ'}{Q} \to -\infty \quad \text{as } r \to +\infty, \quad \text{and}$$

$$Q' < 0,\ Q'' \geqslant 0, \quad \sup_{r>2\rho} \frac{QQ''}{(Q')^2} < +\infty.$$

*Then*

$$\left|u(x)\right| \leqslant C\left(\int_{|x|}^{\infty} \frac{-Q^2}{rQ'}\frac{dr}{r}\right)^{\frac{1}{1-\sigma}}, \quad x \in A_{2\rho,\infty}. \tag{1.15}$$

The last theorem deals with the special case of planar domains.

**Theorem 1.4.** *Let $u$ be a solution to inequality* (1.1) *in $A_{\rho,\infty}$, $\rho > 1$. Let $Q(\cdot):(\rho,+\infty)\to\mathbb{R}^+$ be such that*

$$Q(r) \leqslant \inf_{|x|=r} c(x) \quad \text{and} \qquad \sup_{r>\rho}\left|\frac{Q' r \ln r}{Q}\right| < +\infty.$$

*Let*

$$A(x) \equiv 2.$$

(a) *Suppose that*

$$\int_{\rho}^{+\infty} Q \ln r \frac{dr}{r} = +\infty.$$

*Then*

$$|u(x)| \leqslant C\left(\int_{\rho}^{|x|} Q \ln r \frac{dr}{r}\right)^{\frac{1}{1-\sigma}}, \quad |x| > 2\rho. \tag{1.16}$$

(b) *Suppose that*

$$\int_{\rho}^{+\infty} Q \ln r \frac{dr}{r} < +\infty.$$

*Then*

$$\left|u(x)\right| \leqslant C\left(\int_{|x|}^{+\infty} Q \ln r \frac{dr}{r}\right)^{\frac{1}{1-\sigma}}, \quad |x| > 2\rho. \tag{1.17}$$

**Remark 1.5.** As the results for the exterior of the ball and for the punctured ball are essentially the same, we give only the proof for the former case. Let us note that the assumption of $c(x)$ being locally *strictly* positive is not necessary. We could easily deal with the case of $c(x)$ being only nonnegative and the same results as given here hold. We do not pursue this issue only to avoid unnecessary complication in the notation. For the same reason we confine ourselves to the case of the exterior of the ball—we can easily replace $A_{\rho,\infty}$ in the theorems and their proofs by a domain $\Omega \subset A_{\rho,\infty}$ imposing additional condition $u\,|_{\partial\Omega\cap A_{\rho,\infty}} = 0$. Moreover, it can be easily seen from the method of our proof that we may omit the requirement of *uniform* ellipticity and demand that $\{a_{ij}(x)\}$ be only *locally* uniformly elliptic and its elements be uniformly bounded, thus allowing for degenerate at infinity (or 0) operators.

Theorems 1.1 and 1.2 are similar and the general idea is contained in Theorem 1.1. Theorem 1.2 shows how to deal with non-smooth functions through the means of an appropriate regularisation. Further, we note that the results of Theorems 1.1–1.4 in case (a) imply the uniqueness of solutions of the first boundary value problem for (1.1) in $A_{\rho,\infty}$.

Now we provide some examples which illustrate the power of our results.

**Example 1.6.** Natural examples of functions $Q$ which satisfy the conditions of Theorem 1.1(a) can be given by

$$Q_{k,\epsilon}(r) = \left(\prod_{j=1}^{k-1} \underbrace{\ln\ldots\ln}_{j} r\right)^{-1} \cdot (\underbrace{\ln\ldots\ln}_{k} r)^{-\varepsilon}, \quad k \geqslant 1,\ 1 > \varepsilon \geqslant 0.$$

One can easily check that for $Q_{k,\epsilon}$ estimate (1.10) reads as follows

$$\left|u(x)\right| \leqslant C(\underbrace{\ln\ldots\ln}_{k} r)^{\frac{1-\varepsilon}{1-\sigma}}.$$

**Example 1.7.** If, on the other hand, we choose

$$Q(r) = (\ln r)^{\alpha}, \quad \alpha > 0,$$

estimate (1.10) yields

$$\left|u(x)\right| \leqslant C(\ln r)^{\frac{1+\alpha}{1-\sigma}},$$

which is better than (1.9).

**Example 1.8.** Next, if we choose $Q$ such that the conditions of Theorem 1.1(b) are satisfied, a rather natural example is given by

$$Q_{\epsilon}(r) = r^{-\varepsilon}, \quad \varepsilon > 0,$$

then it is easy to see that the equation

$$\Delta u = \frac{1}{|x|^{2+\varepsilon}} u^{\sigma}$$

admits a solution growing at infinity

$$u_{\epsilon}(x) = C_{\varepsilon}|x|^{\frac{\varepsilon}{\sigma-1}}.$$

Note that it has the same order of growth as predicted by our estimate.

**Remark 1.9.** If $Q$ has the form $r^{\epsilon} f(r)$, $\epsilon \neq 0$, $f'(r) = o(\frac{f}{r})$ as $r \to \infty$ one can easily verify that the estimates (1.10), (1.11) coincide with (1.9). So, the estimate provided by Theorem 1.1 presents the main interest when $Q$ has the form

$$\prod_{j=1}^{k} (\underbrace{\ln\ldots\ln}_{j} r)^{\alpha_j}.$$

Now we provide an example which demonstrates the sharpness of estimate (1.10).

**Example 1.10.** Consider the function $u = (\ln r)^{\frac{1}{1-\sigma}}$ which solves the equation

$$\Delta u - \frac{1}{2}\sum_{j=1}^{2}\frac{x_j u_{x_j}}{|x|^2} = |x|^{-2}\left(\frac{1}{2(\sigma-1)} + \frac{\sigma}{(\sigma-1)^2}\frac{1}{\ln r}\right)u^\sigma$$

in $A_{2,\infty} \subset \mathbb{R}^2$. One can easily check that estimate (1.10) provides the right answer.

On the other hand, if $c(x) \to \infty$ fast enough (faster then any power of $|x|$) then the scaling argument, as well as estimates (1.10), (1.11) fail to produce the right answer.

**Example 1.11.** Indeed, set $u(x) = e^{-|x|}$. Then $u$ solves the equation

$$\Delta u - (n-1)\sum_{i=1}^{n}\frac{x_i u_{x_i}}{|x|^2} = \frac{e^{(\sigma-1)|x|}|x|^2}{|x|^2}u^\sigma$$

in $A_{1,\infty}$. The scaling argument provides the estimate

$$\left|u(x)\right| \leqslant Ce^{-|x|/\delta}, \quad \delta > 1.$$

The estimate (1.10) yields

$$\left|u(x)\right| \leqslant C|x|^{\frac{1}{1-\sigma}}e^{-|x|}$$

which is obviously false, and one can easily check that estimate (1.14) gives the right answer

$$\left|u(x)\right| \leqslant Ce^{-|x|}.$$

The same effect takes place when $c(x) \to 0$ very fast (faster than any power of $|x|$), as is demonstrated by the following example.

**Example 1.12.** Set $u(x) = e^{|x|}$ in $A_{1,\infty}$, which solves the equation

$$\Delta u - (n-1)\sum_{i=1}^{n}\frac{x_i u_{x_i}}{|x|^2} = \frac{e^{(1-\sigma)|x|}|x|^2}{|x|^2}u^\sigma.$$

The estimate provided by the scaling argument is

$$\left|u(x)\right| \leqslant Ce^{|x|\delta}, \quad \delta > 1,$$

whereas estimate (1.11) yields

$$\left|u(x)\right| \leqslant C|x|^{\frac{1}{1-\sigma}}e^{|x|},$$

which is clearly not true. Like in the previous example, one can easily check that estimate (1.15) provides the correct result

$$\left|u(x)\right| \leqslant Ce^{|x|}.$$

Theorem 1.4 describes the special case of planar domains. This situation arises when we study the equation of the form

$$\Delta u = \frac{c(x)}{|x|^2}u^\sigma, \quad x \in A_{\rho,\infty} \subset \mathbb{R}^2,$$

and $c(x)$ behaves like a product of logarithmic functions depending on radius. In the three examples below one can easily check that the estimates given by Theorem 1.4 yield the right answer.

**Example 1.13.** Let us consider the function $u = (\ln\ln r)^{\frac{1}{1-\sigma}}$ in $A_{10,\infty} \subset \mathbb{R}^2$, which solves the equation

$$\Delta u = \left(\frac{1}{\sigma-1}(\ln r)^{-2} + \frac{\sigma}{(1-\sigma)^2}(\ln r)^{-2}(\ln\ln r)^{-1}\right)\frac{u^\sigma}{r^2}.$$

Estimate (1.10) gives only

$$\left|u(x)\right| \leqslant C(\ln r)^{\frac{1}{\sigma-1}}, \quad |x| > 20,$$

which allows for solutions growing at infinity.

**Example 1.14.** This time we take the function $u = (\ln r)^{\frac{1}{1-\sigma}}$, which solves the equation

$$\Delta u = \frac{\sigma}{(1-\sigma)^2}\frac{1}{r^2 \ln r}u^{\sigma} \quad \text{in } A_{2,\infty} \in \mathbb{R}^2.$$

If we apply Theorem 1.1 we will be left with the estimate

$$\left|u(x)\right| \leqslant C(\ln\ln r)^{\frac{1}{1-\sigma}}, \quad |x| > 4,$$

which is again far from optimal.

**Example 1.15.** Let us choose a number $k$ such that $k+1 > \sigma$. Then the function $u = (\ln r)^{\frac{k}{\sigma-1}}$ solves the equation

$$\Delta u = \frac{k}{\sigma-1}\frac{k+1-\sigma}{\sigma}(\ln r)^{\frac{-k-2}{\sigma-1}}\frac{u^{\sigma}}{r^2}$$

in $A_{2,+\infty} \subset \mathbb{R}^2$. The estimate provided by Theorem 1.1 reads as

$$\left|u(x)\right| \leqslant \left(\ln|x|\right)^{\frac{k+1}{\sigma-1}}, \quad |x| \geqslant 4,$$

and we again observe that in this case we roughly speaking lose one logarithm.

## 2. Auxiliary facts and central lemma

### 2.1. Some more notation

For a constant $\gamma > 1$ we introduce functions $c_\gamma(\cdot) : (\rho/\gamma, +\infty) \to \mathbb{R}^+$ defined by

$$c_\gamma(r) = \inf_{x \in \bar{A}_{r/\gamma, r\gamma}} c(x).$$

For a constant $\gamma > 1$ and a function $f : (\rho/\gamma, +\infty) \to \mathbb{R}^+$ we define a function $S_\gamma[f] : (\rho, +\infty) \to \mathbb{R}^+$ by

$$S_\gamma[f](r) = \int_{r/\gamma}^{r\gamma} f(s)\frac{ds}{s}.$$

It is clear that if for any $r > \rho/\gamma$ $0 \leqslant f(r) \leqslant c_\gamma(r)$ then $S_\gamma[f](r) \leqslant 2(\ln\gamma)\inf_{|x|=r} c(x)$ and $|\frac{d}{dr}S_\gamma[f](r)| \leqslant \gamma\frac{\inf_{|x|=r} c(x)}{r}$.

We also note that we often understand a function defined on a subset of positive real semiaxis as a function defined on a subset of $\mathbb{R}^n$. In this case we naturally define $f(x) = f(|x|)$.

Since for technical purposes we will need $q(r)$ to be defined for $r > \rho/\delta$ we set $q(r) = \min(\inf_{\rho \leqslant |x| \leqslant r\delta} c(x))$ for $r \in (\rho/\delta, \rho)$.

When we speak about inf or sup of a function on some set $X$ we mean the intersection of $X$ with the domain of definition of this function.

### 2.2. Maximum principle

In this paper we use a maximum principle in the following form:

**Proposition 2.1** *(Maximum principle). Let $\Omega$ be a bounded domain and $u$, $v$ are a solution and a supersolution to* (1.1) *in $\Omega$, respectively. If*

$$u_+|_{\partial\Omega} \leqslant v_+|_{\partial\Omega} \quad (u_-|_{\partial\Omega} \leqslant v_-|_{\partial\Omega}),$$

*then*

$$u_+ \leqslant v_+ \quad (u_- \leqslant v_-) \text{ in } \Omega.$$

**Remark 2.2.** If two functions $\phi, \psi \in C(\bar{\Omega})$ then there is no problem in understanding the relation $\phi|_{\partial\Omega} \leqslant \psi|_{\partial\Omega}$. If they are only $C(\Omega)$ then we understand it in the following standard sense:

$$\limsup_{x \to \partial\Omega}\left(\phi(x) - \psi(x)\right) \leqslant 0.$$

**Corollary 2.3.** *Let $u$ and $v$ be a solution and a supersolution to* (1.1) *in $A_{\rho,\infty}$ and*

$$u, v \to 0 \quad \text{as } x \to \infty.$$

*Then the maximum principle holds in $A_{\rho,\infty}$: if*

$$u_+|_{\partial A_{\rho,\infty}} \leqslant v_+|_{\partial A_{\rho,\infty}} \quad (u_-|_{\partial A_{\rho,\infty}} \leqslant v_-|_{\partial A_{\rho,\infty}}),$$

*then*

$$u_+ \leqslant v_+ \quad (u_- \leqslant v_-) \text{ in } A_{\rho,\infty}.$$

**Proof.** We first apply the maximum principle to the annulus $A_{\rho,\rho_1}$, $\rho < \rho_1 < +\infty$ from which we have

$$u_+(x) \leqslant v_+(x) + \max(u_+, v_+)|_{|x|=\rho_1}$$

in $A_{\rho,\rho_1}$. Now since the last term vanishes as $\rho_1 \to +\infty$, we obtain

$$u_+ \leqslant v_+ \quad \text{in } A_{\rho,\infty}. \qquad \square$$

In the following lemma we may assume in the proof that $\sup_{x\in\partial A_{\rho_1,\rho_2}} |u(x)| < +\infty$. If it is not the case then we first apply this lemmas to a slightly smaller annulus $A_{\rho_1+\varepsilon,\rho_2-\varepsilon}$ and then pass to the limit as $\varepsilon \to 0$.

**Lemma 2.4.** *Let $u$ be a solution to inequality* (1.1) *in $A_{\rho_1,\rho_2}$. Let the functions $R_1(r)$, $R_2(r)$ be given by*:

$$R_1(r) = \int_{\rho_1}^{r} V(s)\frac{ds}{s}, \qquad R_2(r) = \int_{r}^{\rho_2} V(s)\frac{ds}{s},$$

*where $V(s) \geqslant 0$; $s \in (\rho_1, \rho_2)$ and $V' \in L_{\infty,\mathrm{loc}}(\rho_1, \rho_2)$.*

*Then in $A_{\rho_1,\rho_2}$ the following estimate holds*

$$\sup_{|x|=r} |u(x)| \leqslant \frac{\sigma+1}{\sigma-1} \max\bigl(C_1(r) R_1^{\frac{2}{1-\sigma}}(r), C_2(r) R_2^{\frac{2}{1-\sigma}}(r)\bigr), \tag{2.1}$$

*where*

$$C_1(r) = \left[\sup_{x\in A_{\rho_1,r}} \frac{2\Phi}{\sigma-1}\cdot\left(\frac{\sigma+1}{\sigma-1}\frac{V^2}{c(x)} + R_1\left(\left(2-A(x)\right)\frac{V}{c(x)} - \frac{rV'}{c(x)}\right)_+\right)\right]^{\frac{1}{\sigma-1}},$$

$$C_2(r) = \left[\sup_{x\in A_{r,\rho_2}} \frac{2\Phi}{\sigma-1}\cdot\left(\frac{\sigma+1}{\sigma-1}\frac{V^2}{c(x)} + R_2\left(\left(A(x)-2\right)\frac{V}{c(x)} + \frac{rV'}{c(x)}\right)_+\right)\right]^{\frac{1}{\sigma-1}}.$$

**Proof.** First let us recall that for a radial function $f(r)$ the expression for $\mathcal{L}f(r)$ takes the following form

$$\mathcal{L}f(r) = \Phi f'' + \frac{T-\Phi}{r} f'. \tag{2.2}$$

Thus, if we want a positive function $f(r)$ to be a supersolution to inequality (1.1) it must satisfy the inequality

$$\Phi f'' + \frac{T-\Phi}{r} f' \leqslant \frac{c(x)}{r^2} f^\sigma \quad \text{in } A_{\rho_1,\rho_2}. \tag{2.3}$$

Let us introduce the function

$$f(x,y) = x^{\frac{2}{1-\sigma}} - \frac{2}{1-\sigma} y^{\frac{2}{1-\sigma}-1}(x-y) + \frac{2}{\sigma-1} y^{\frac{2}{1-\sigma}}.$$

It is clear that

$$f_x'(x,x) = 0, \qquad f(x,x) = \frac{\sigma+1}{\sigma-1} x^{\frac{2}{1-\sigma}},$$
$$f(x,y) \to +\infty \quad \text{as } x \to 0+,$$
$$f(x,y) \geqslant x^{\frac{2}{1-\sigma}}, \quad 0 \leqslant x \leqslant y,$$
$$f_x'(x,y) = \frac{2}{1-\sigma}\bigl(x^{\frac{2}{1-\sigma}-1} - y^{\frac{2}{1-\sigma}-1}\bigr) \quad \text{and} \quad f_{xx}''(x,y) = \frac{2}{1-\sigma}\left(\frac{2}{1-\sigma}-1\right) x^{\frac{2\sigma}{1-\sigma}}.$$

Let us fix some number $\xi \in (\rho_1, \rho_2)$ and denote

$$g_{1,\xi}(r) = f\big(R_1(r), R_1(\xi)\big), \qquad g_{2,\xi}(r) = f\big(R_2(r), R_2(\xi)\big).$$

Let us show that for $C \geqslant C_1(\xi)$ the function $Cg_{1,\xi}$ is a supersolution to inequality (1.1) in $A_{\rho_1,\xi}$. Indeed, for such values of $C$,

$$\begin{aligned}\mathcal{L}Cg_{1,\xi} &= C\frac{R_1^{\frac{2\sigma}{1-\sigma}}}{r^2}\frac{2}{\sigma-1}\left[\frac{\sigma+1}{\sigma-1}\Phi V^2 + R_1\left(1-\left(\frac{R_1(\xi)}{R_1}\right)^{\frac{2}{1-\sigma}-1}\right)\big((2\Phi - T)V - \Phi V' r\big)\right] \\ &\leqslant \frac{c(x)}{r^2} g_{1,\xi}^{\sigma} \cdot C_1^{\sigma-1} C \leqslant \frac{c(x)}{r^2}(Cg_{1,\xi})^{\sigma}.\end{aligned}$$

Analogously, the function $Cg_{2,\xi}$ is a supersolution to inequality (1.1) in $A_{\xi,\rho_2}$ for the values $C \geqslant C_2(\xi)$. Let us introduce now the function

$$F_\xi(r) = \begin{cases} k_1 g_{1,\xi}, & r \leqslant \xi, \\ k_2 g_{2,\xi}, & r \geqslant \xi, \end{cases}$$

where the constants $k_1, k_2$ satisfy the conditions

$$\begin{aligned}&k_1 g_{1,\xi}(\xi) = k_2 g_{2,\xi}(\xi), \\ &k_1 \geqslant C_1(\xi), \quad k_2 \geqslant C_2(\xi).\end{aligned}$$

It is obvious that these conditions are satisfied if we choose

$$\begin{aligned}&k_1 = \alpha R_1^{\frac{2}{\sigma-1}}(\xi), \quad k_2 = \alpha R_2^{\frac{2}{\sigma-1}}(\xi), \\ &\alpha \geqslant \max\big(C_1(\xi) R_1^{\frac{2}{1-\sigma}}(\xi), C_2(\xi) R_2^{\frac{2}{1-\sigma}}(\xi)\big).\end{aligned}$$

Now, $F_\xi(\xi) = \frac{\sigma+1}{\sigma-1}\alpha$, and $F$ is a supersolution to (1.1) in $A_{\rho_1,\rho_2}$. From our construction it is clear that

$$F(r) \to +\infty \quad \text{as } r \to \rho_1 + 0,\ r \to \rho_2 - 0.$$

We can therefore apply the maximum principle to $u$ and $F$ and immediately obtain the assertion of this lemma. □

In order to move further we need to prove several auxiliary statements.

The proof of the next proposition is very simple and is a direct application of Fubini's theorem, and so we omit it.

**Proposition 2.5.** *Let $f : (\rho/\gamma, +\infty) \to \mathbb{R}^+$ for some $\gamma > 1$. Then $\int^{+\infty} S_\gamma[f](r)\frac{dr}{r}$ and $\int^{+\infty} f(r)\frac{dr}{r}$ converge or diverge simultaneously and*

$$\int_\rho^r S_\gamma[f](s)\frac{ds}{s} \geqslant \min(\ln\gamma, \ln\varepsilon)\int_\rho^r f(s)\frac{ds}{s}, \quad r \geqslant \varepsilon\rho,$$

$$\int_r^{+\infty} S_\gamma[f](s)\frac{ds}{s} \geqslant \ln\gamma \int_r^{+\infty} f(s)\frac{ds}{s}, \quad r \geqslant \rho.$$

**Proposition 2.6.**

$$S_\delta[q](r) \leqslant \frac{1+K}{\ln\delta}\int_\rho^r S_\delta[q](s)\frac{ds}{s}, \quad r > \rho\delta,$$

*where $K = \sup_{r>\rho}\frac{q(r\delta)}{q(r)}$.*

**Proof.**

$$S_\delta[q](r) = \int_{r/\delta}^{r\delta} q(s)\frac{ds}{s} \leqslant \int_\rho^r q(s)\frac{ds}{s} + \int_r^{r\delta} q(s)\frac{ds}{s} \leqslant (1+K)\int_\rho^r q(s)\frac{ds}{s} \leqslant \frac{1+K}{\ln\delta}\int_\rho^r S_\delta[q](s)\frac{ds}{s}.$$

We give the next proposition without proof as it is similar to the one above. □

**Proposition 2.7.**

$$S_\delta[q](r) \leqslant \frac{1+K}{\ln\delta} \int_r^{+\infty} S_\delta[q](s)\frac{ds}{s}, \quad r > \rho\delta,$$

*where* $K = \sup_{r>\rho} \frac{q(r)}{q(r\delta)}$.

Now we are ready to pass on to the proof of our theorems.

## 3. Proof of Theorems 1.1–1.4

For a given function $V$ which will be defined separately for each case, we will denote

$$R_1(r) = \int_\rho^r V(s)\frac{ds}{s}, \qquad R_2(r) = \int_r^{\rho_1} V(s)\frac{ds}{s},$$

$$R_{2,\rho_1}(r) = \int_r^{\rho_1} V(s)\frac{ds}{s}, \qquad I = \int_\rho^\infty V(s)\frac{ds}{s}.$$

In the proof below the letter $C$ stands for constants independent on $\xi$, $\rho_1$.

First, we set $V = Q$ in case I, $V = S_\delta[q]$ in case II, $V = |\frac{Q^2}{rQ'}|$ in case III, $V = Q\ln r$ in case IV. Note that in each case except IV

$$k_1 := \sup_{r>\rho} \frac{V}{c(x)} < +\infty.$$

(In the cases I and III, $k_1 = 1$, and in the case II, $k_1 \leqslant 2\ln\delta$.) In case IV we set $k_1 = 0$.

In cases labeled by (a)

$$R_{2,\rho_1}(r) \to +\infty \quad \text{as } \rho_1 \to \infty,$$

$$R_1(r) \to +\infty \quad \text{as } r \to \infty.$$

In cases labeled by (b) $I < +\infty$ and

$$R_{2,\rho_1}(r) \to R_2(r) \quad \text{as } \rho_1 \to +\infty,$$

$$R_1(r) \to I \quad \text{as } r \to \infty,$$

$$R_2(r) \to 0 \quad \text{as } r \to \infty,$$

$$R_1(r), R_{2,\rho_1}(r) \leqslant I \quad \text{for all } r, \rho_1 \geqslant \rho.$$

Let us fix some number $\xi > 2\rho$. We start with applying Lemma 2.4 to the annulus $A_{\rho,\rho_1}$, $\rho_1 > \xi$. It immediately yields the following estimate

$$M(\xi) \leqslant \frac{\sigma+1}{\sigma-1}\max\bigl(C_1(\xi)R_1^{\frac{2}{1-\sigma}}(\xi), C_{2,\rho_1}(\xi)R_{2,\rho_1}^{\frac{2}{1-\sigma}}(\xi)\bigr), \quad \xi \in (\rho,\rho_1), \tag{3.1}$$

where

$$C_1(\xi) = \left[\sup_{x\in A_{\rho,\xi}} \frac{2\Phi}{\sigma-1}\cdot\left(\frac{\sigma+1}{\sigma-1}\frac{V^2}{c(x)} + \left(\bigl(2-A(x)\bigr)\frac{V}{c(x)} - \frac{rV'}{c(x)}\right)_+ R_1\right)\right]^{\frac{1}{\sigma-1}},$$

$$C_{2,\rho_1}(\xi) = \left[\sup_{x\in A_{\xi,\rho_1}} \frac{2\Phi}{\sigma-1}\cdot\left(\frac{\sigma+1}{\sigma-1}\frac{V^2}{c(x)} + \left(\bigl(A(x)-2\bigr)\frac{V}{c(x)} + \frac{rV'}{c(x)}\right)_+ R_{2,\rho_1}\right)\right]^{\frac{1}{\sigma-1}}.$$

Let us denote

$$f_1(r) = \sup_{|x|=r}\left|\frac{rV'}{c(x)}\right|, \qquad f_2(r) = \sup_{|x|=r}\frac{V^2(r)}{c(x)}.$$

We claim that in each case

$$k_2 := \sup_{r>\rho} f_1(r) < +\infty.$$

Indeed, for case I it is contained in the condition of the theorem, since in this case $V = Q$ and

$$f_1(r) \leqslant \left|\frac{rQ'}{Q}\right|.$$

In case II it follows from the property of $S_\delta[q]$:

$$\left|\frac{d}{dr} S_\delta[q]\right| \leqslant \delta \frac{\inf_{|x|=r} c(x)}{r}.$$

In case III we see that

$$f_1(r) \leqslant \left|\frac{rV'}{Q}\right| = \left|2 - \frac{Q}{rQ'} - \frac{QQ''}{(Q')^2}\right|,$$

which is uniformly bounded in $A_{\rho,\infty}$ according to the condition of the theorem. (Without loss of generality we can assume that $|Q'(\rho)| > 0$ since otherwise we could apply our reasoning to slightly smaller annulus in which $|Q'| > \text{const} > 0$.)

In case IV we evaluate

$$f_1(r) \leqslant \left|1 + \frac{Q' r \ln r}{Q}\right|,$$

which is again uniformly bounded in $A_{\rho,\infty}$.

Second, we claim that in cases labeled by (a) there exist $k_3, k_4 > 0$ such that

$$f_2(r) \leqslant k_3 + k_4 R_1(r),$$

and in cases labeled by (b) there exists a constant $k_4 > 0$ such that

$$f_2(r) \leqslant k_4 R_2(r).$$

In case I we estimate first

$$f_2(r) \leqslant Q(r)$$

and note that $|Q'(r)| \leqslant CQr^{-1}$. Our statement follows then from the Newton–Leibnitz formula.

In case II we estimate first

$$f_2(r) \leqslant 2 \ln \delta S_\delta[q](r)$$

and use Propositions 2.6 and 2.7 to obtain the desired estimate.

In case III we write first

$$f_2(r) \leqslant \frac{Q^3}{r^2 (Q')^2}$$

and explicitly calculating the derivative of the last expression we see that

$$\left(\frac{Q^3}{r^2(Q')^2}\right)' = \frac{3Q^2}{r^2 Q'} - \frac{2Q^3}{r^3 (Q')^2} - \frac{2Q^3 Q''}{(Q')^3},$$

which is $\leqslant 3(R_1)'$ in case III(a). In case III(b) we write it as

$$\frac{Q^2}{r^2 Q'}\left(3 - \frac{2Q}{rQ'} - \frac{2QQ''}{(Q')^2} Q'\right),$$

which shows that it is $\geqslant C R_2'(r)$ with some constant $C$. (We again assume in this place that $|Q'(\rho)| > 0$.)

In case IV

$$f_2(r) \leqslant Q(\ln r)^2$$

and estimating the derivative of the last expression we have

$$\left|\frac{d}{dr} Q(\ln r)^2\right| = \left|Q'(\ln r)^2 + 2\frac{Q \ln r}{r}\right| \leqslant C \frac{Q \ln r}{r} = C\big(R_1(r)\big)' = -C\big(R_2(r)\big)'$$

with some constant $C$. Application of Newton–Leibnitz formula finishes the proof.

Now, in cases labeled as (a) we estimate

$$C_1(\xi) \leqslant C(\sigma, \nu_2) \sup_{\rho<\tau<\xi} \big(k_3 + (k_1 + k_2 + k_4) R_1(\tau)\big)^{\frac{1}{\sigma-1}} \leqslant C R_1(\xi), \quad \xi \geqslant 2\rho,$$

$$C_{2,\rho_1}(\xi) \leqslant C(\sigma, \nu_2) \sup_{\xi<\tau<\rho_1} \big(k_3 + k_4\big(R_1(\xi) + R_{2,\tau}(\xi)\big) + (k_1 + k_2) R_{2,\rho_1}(\tau)\big)^{\frac{1}{\sigma-1}} \leqslant C\big(R_1(\xi) + R_{2,\rho_1}(\xi)\big)^{\frac{1}{\sigma-1}}.$$

Estimate (3.1) now reads as

$$M(\xi) \leqslant C \max\big(R_1^{\frac{1}{1-\sigma}}(\xi), R_{2,\rho_1}^{\frac{2}{1-\sigma}}(\xi)\big(R_1(\xi) + R_{2,\rho_1}(\xi)\big)^{\frac{1}{\sigma-1}}\big).$$

Sending $\rho_1$ to $\infty$ eliminates the second term and finishes the proof.

In case (b) we estimate

$$C_1(\xi) \leqslant C(\sigma, \nu_2) \sup_{\rho<\tau<\xi} \big(k_4 R_2(\tau) + (k_1+k_2)R_1(\tau)\big)^{\frac{1}{\sigma-1}} \leqslant C I^{\frac{1}{\sigma-1}},$$

$$C_{2,\rho_1}(\xi) \leqslant C(\sigma, \nu_2) \sup_{\xi<\tau<\rho_1} \big(k_4 R_2(\tau) + (k_1+k_2)R_{2,\rho_1}(\tau)\big)^{\frac{1}{\sigma-1}} \leqslant C\big(R_2(\xi) + R_{2,\rho_1}(\xi)\big)^{\frac{1}{\sigma-1}} \leqslant C R_2^{\frac{1}{\sigma-1}}(\xi).$$

Estimate (3.1) now reads as

$$M(\xi) \leqslant C \max\big(I^{\frac{1}{\sigma-1}} R_1^{\frac{2}{1-\sigma}}(\xi),\ R_2^{\frac{1}{\sigma-1}}(\xi) R_{2,\rho_1}^{\frac{2}{1-\sigma}}(\xi)\big).$$

Sending $\rho_1$ to $\infty$ and observing that

$$R_1(\xi) \geqslant \text{const} > 0, \quad \xi \geqslant 2\rho,$$

finishes the proof. □

## 4. Results for the punctured ball

In this section we collect the results analogous to those we have obtained for the exterior domain. We do not give proofs here as they repeat those for the preceding case.

**Theorem 4.1.** *Let $u$ be a solution to inequality* (1.1) *in $A_{0,\rho}$. Let $Q(\cdot) : (0,\rho) \to \mathbb{R}^+$ be such that*

$$Q(r) \leqslant \inf_{|x|=r} c(x),$$

$$\sup_{r<\rho} \left|\frac{Q' r}{Q}\right| < +\infty.$$

(a) *If $\int_0^\rho Q(r)\frac{dr}{r} = +\infty$, then*

$$|u(x)| \leqslant C\left(\int_{|x|}^{\rho} Q(r)\frac{dr}{r}\right)^{\frac{1}{1-\sigma}}, \quad x \in A_{0,\rho/2}, \tag{4.1}$$

*and as a consequence*

$$u(x) \to 0 \quad \text{as } x \to 0.$$

(b) *If $\int_0^\rho Q(r)\frac{dr}{r} < +\infty$ and $Q$ is bounded, then*

$$|u(x)| \leqslant C\left(\int_0^{|x|} Q(r)\frac{dr}{r}\right)^{\frac{1}{1-\sigma}}, \quad x \in A_{0,\rho/2}. \tag{4.2}$$

**Theorem 4.2.** *Let $u$ be a solution to inequality* (1.1) *in $A_{0,\rho}$. Let $q(\cdot) : (0,\rho) \to \mathbb{R}^+$ be such that*

$$q(r) \leqslant \inf_{x\in \bar{A}_{r/\delta, r\delta}} c(x) \quad \text{for some } \delta > 1.$$

(a) *If $\int_0^\rho q(r)\frac{dr}{r} = +\infty$ and*

$$\sup_{r<\rho} \frac{q(r/\delta)}{q(r)} < +\infty,$$

*then*

$$|u(x)| \leqslant C\left(\int_{|x|}^{\rho} q(r)\frac{dr}{r}\right)^{\frac{1}{1-\sigma}}, \quad x \in A_{0,\rho/2}. \tag{4.3}$$

(b) *If* $\int_0^\rho q(r)\frac{dr}{r} < +\infty$, *q is bounded and*

$$\sup_{r<\rho/\delta} \frac{q(r\delta)}{q(r)} < +\infty,$$

*then*

$$\left|u(x)\right| \leqslant C\left(\int_0^{|x|} q(r)\frac{dr}{r}\right)^{\frac{1}{1-\sigma}}, \quad x \in A_{0,\rho/2}. \tag{4.4}$$

**Theorem 4.3.** *Let u be a solution to inequality* (1.1) *in* $A_{\rho,\infty}$. *Let* $Q(\cdot) : (0,\rho) \to \mathbb{R}^+$ *be such that*

$$Q(r) \leqslant \inf_{|x|=r} c(x).$$

(a) *Suppose that*

$$\frac{rQ'}{Q} \to -\infty \quad \text{as } r \to 0, \quad \text{and}$$

$$Q' \leqslant 0,\ Q'' \geqslant 0, \quad \sup_{r<\rho} \frac{Q\,Q''}{(Q')^2} < +\infty.$$

*Then*

$$u(x) \to 0 \quad \text{as } x \to 0,$$

*and*

$$\left|u(x)\right| \leqslant C\left(\int_{|x|}^{\rho} \frac{-Q^2}{rQ'}\frac{dr}{r}\right)^{\frac{1}{1-\sigma}}, \quad x \in A_{0,\rho/2}. \tag{4.5}$$

(b) *Suppose that*

$$\frac{rQ'}{Q} \to +\infty \quad \text{as } r \to 0, \quad \text{and}$$

$$Q' > 0,\ Q'' \geqslant 0, \quad \sup_{r<\rho} \frac{Q\,Q''}{(Q')^2} < +\infty.$$

*Then*

$$\left|u(x)\right| \leqslant C\left(\int_0^{|x|} \frac{Q^2}{rQ'}\frac{dr}{r}\right)^{\frac{1}{1-\sigma}}, \quad x \in A_{0,\rho/2}. \tag{4.6}$$

**Theorem 4.4.** *Let u be a solution to inequality* (1.1) *in* $A_{0,\rho}$, $\rho < 1$. *Let* $Q(\cdot) : (0,\rho) \to \mathbb{R}^+$ *be such that*

$$Q(r) \leqslant \inf_{|x|=r} c(x) \quad \text{and} \quad \sup_{r<\rho}\left|\frac{Q' r \ln r}{Q}\right| < +\infty.$$

*Let*

$$A(x) \equiv 2.$$

(a) *Suppose that*

$$\int_0^{\rho} Q \ln r \frac{dr}{r} = +\infty.$$

*Then*

$$\left|u(x)\right| \leqslant C\left(\int_{|x|}^{\rho} Q \ln\frac{1}{r}\frac{dr}{r}\right)^{\frac{1}{1-\sigma}}, \quad |x| < \rho/2. \tag{4.7}$$

(b) *Suppose that*

$$\int_0^\rho Q \ln r \frac{dr}{r} < +\infty.$$

*Then*

$$\left|u(x)\right| \leqslant C\left(\int_0^{|x|} Q \ln\frac{1}{r}\frac{dr}{r}\right)^{\frac{1}{1-\sigma}}, \quad |x| < \rho/2. \tag{4.8}$$

## 5. Improvement of estimates

Our estimates are aimed at the "worst case" (demonstrated in Example 4)—that is, $A(x) < 2$ in $A_{\rho,\infty}$ or $A(x) > 2$ if we consider $A_{0,\rho}$. If we consider $A_{\rho,\infty}$ (or $A_{0,\rho}$) and

$$\liminf_{x\to\infty} A(x) > 2 \quad \bigl(\limsup_{x\to 0} A(x) < 2, \text{ respectively}\bigr)$$

then the estimates provided by Theorems 1.1 (4.1, respectively) can be improved. That is the reason why in our examples we used the drift term, which shifts the "effective dimension."

Using the expression (2.2) for the operator $\mathcal{L}$ acting on radial function one can easily verify that both functions (if they are defined)

$$f_1(r) = \int_r^{+\infty} \exp\left(\int_\rho^s \sup_{|x|=\xi}\bigl(2 - A(x)\bigr)\frac{d\xi}{\xi}\right)\frac{ds}{s}, \tag{5.1}$$

$$f_2(r) = \int_0^r \exp\left(\int_s^\rho \sup_{|x|=\xi}\bigl(A(x) - 2\bigr)\frac{d\xi}{\xi}\right)\frac{ds}{s} \tag{5.2}$$

are positive solutions to the inequality $\mathcal{L}u \leqslant 0$ in $A_{\rho,\infty}$ and $A_{0,\rho}$, respectively [18] and hence positive supersolutions to inequality (1.1) in the same domain. It is clear that $f_1(r) \to 0$ as $r \to \infty$ and $f_2(r) \to 0$ as $r \to 0$.

The example of $f_1$ is a fundamental solution to $\Delta u = 0$ in dimension $n \geqslant 3$, which is $c_n r^{2-n}$. The example of $f_2$ is a function $f(r) = r$, which gives a solution to $\Delta u - (n-1)\sum_{j=1}^n \frac{x_j u_{x_j}}{|x|^2}$.

Then,

$$f_3(r) = \int_\rho^r \exp\left(\int_\rho^s \inf_{|x|=\xi}\bigl(2 - A(x)\bigr)\frac{d\xi}{\xi}\right)\frac{ds}{s}, \tag{5.3}$$

$$f_4(r) = \int_r^\rho \exp\left(\int_s^\rho \inf_{|x|=\xi}\bigl(A(x) - 2\bigr)\frac{d\xi}{\xi}\right)\frac{ds}{s} \tag{5.4}$$

are positive solutions to the inequality $\mathcal{L}u \leqslant 0$ in $A_{\rho,\infty}$ and $A_{0,\rho}$ respectively and hence positive supersolutions to inequality (1.1) in the same domains. If the integrals defining them diverge then we have a supersolution tending to $+\infty$ as $r \to \infty$ or $r \to 0$.

The example for both $f_3$, $f_4$ is provided by the fundamental solution to $\Delta u = 0$ in $\mathbb{R}^2$.

It is easy to see that the function $f_1$ is defined if $\liminf_{x\to\infty} A(x) > 2$ and the function $f_2$ is defined if $\limsup_{x\to 0} A(x) < 2$. If, on the other hand, we have $\limsup_{x\to\infty} A(x) < 2$ ($\liminf_{x\to 0} A(x) > 2$) we have a supersolution $f_3(r)$ (respectively $f_4(r)$) which tends to $+\infty$ as $x$ goes to $\infty$ (respectively 0).

To avoid unnecessary "generality" we demonstrate on the simple examples how the estimates can be improved (or established) in several interesting cases. This exposition is restricted for simplicity to the case of the Laplacian. It is readily seen that these results can be easily carried over to general operators of the form (1.2), with coefficients which are Dini continuous at 0 or at infinity.

**Example 5.1.** Let $u$ be a (classical) solution to

$$\Delta u = \frac{1}{|x|^2}u^\sigma, \quad x \in A_{\rho,\infty} \subset \mathbb{R}^n, \ n \geqslant 3.$$

Then Theorem 1.1 yields that $u(x) \to 0$ as $x \to \infty$ and using the maximum principle we immediately obtain

$$\left|u(x)\right| \leqslant \frac{\sup_{|x|=2\rho} |u(x)|}{(2\rho)^{2-n}} |x|^{2-n}, \quad x \in A_{2\rho,\infty},$$

which is far better that the estimate $|u(x)| \leqslant C(\ln|x|)^{\frac{1}{1-\sigma}}$ provided by Theorem 1.1.

**Example 5.2.** Let $u$ be a (classical) solution to

$$\Delta u = u^{\frac{n}{n-2}}, \quad x \in A_{0,1} \subset \mathbb{R}^n,\ n \geqslant 3.$$

The number $\frac{n}{n-2}$ is the critical exponent for the equation of this type. Setting $u = r^{2-n} v$ we have

$$\Delta v + 2(2-n)\sum_{j=1}^{n} \frac{x_j}{|x|^2} v_{x_j} = \frac{v^{\frac{n}{n-2}}}{|x|^2}.$$

which immediately yields $v(x) \to 0$ as $x \to 0$ and consequently $u(x) = o(|x|^{2-n})$. Using the standart argument one can easily show that this fact implies the "removability of singularity"—we can define $u(0)$ so that the function $u(x)$ will be a solution in the whole ball $B_1$.

The following example is more subtle because it deals with the equation where the linear potential is present.

**Example 5.3.** Let $u$ be a (classical) solution to

$$\Delta u + \frac{C}{|x|^2} u = |x|^p u^\sigma, \quad x \in A_{1,\infty} \subset \mathbb{R}^n,\ n \geqslant 3,$$

where the constant $C \leqslant C_H = \frac{(n-2)^2}{4}$. Let us denote

$$\lambda_+ = \frac{2-n+\sqrt{(n-2)^2-4C}}{2}, \qquad \lambda_- = \frac{2-n-\sqrt{(n-2)^2-4C}}{2}.$$

It is clear that $\lambda_+ \geqslant \lambda_-$ and the only case when they are equal is when $C = \frac{(n-2)^2}{4}$. Now, one can easily check that the functions $r^{\lambda_+}$, $r^{\lambda_-}$ if $C < \frac{(n-2)^2}{4}$ and $r^{\frac{2-n}{2}}$, $r^{\frac{2-n}{2}} \ln r$ if $C = \frac{(n-2)^2}{4}$ are positive solutions to

$$\Delta u + \frac{C}{|x|^2} u = 0 \quad \text{in } A_{1,\infty}.$$

Let $\lambda$ be $\lambda_+$ or $\lambda_-$. Performing the "ground-state transform" $u = r^\lambda v$ we arrive at the equation

$$\Delta v + 2\lambda \sum_{j=1}^{n} \frac{x_j}{|x|^2} v_{x_j} = |x|^{p+\lambda(\sigma-1)} v^\sigma.$$

Suppose first that $p + \lambda(\sigma - 1) \neq -2$. In this case Theorem 1.1 yields

$$\left|v(x)\right| \leqslant \tilde{C}|x|^{\frac{2+p}{1-\sigma}-\lambda}, \quad |x| > 2,$$

and consequently

$$\left|u(x)\right| \leqslant \tilde{C}|x|^{\frac{2+p}{1-\sigma}}, \quad |x| > 2. \tag{5.5}$$

Now, suppose that $p + \lambda(\sigma - 1) = -2$, or, in a more convenient form, $\lambda = \frac{2+p}{1-\sigma}$. In this case the estimate for $v$ provided by Theorem 1.1 reads as

$$\left|v(x)\right| \leqslant \tilde{C}\big(\ln|x|\big)^{\frac{1}{1-\sigma}}, \quad |x| > 2,$$

and thus

$$\left|u(x)\right| \leqslant \tilde{C}|x|^{\frac{2+p}{1-\sigma}}\big(\ln|x|\big)^{\frac{1}{1-\sigma}}, \quad |x| > 2. \tag{5.6}$$

Thus, if $\lambda_+$ or $\lambda_-$ is equal to $\frac{2+p}{1-\sigma}$ then $u(x) = o(|x|^{\frac{2+p}{1-\sigma}})$ as $x \to \infty$.

## Acknowledgments

The author would like to thank Professor V.A. Kondratiev for valuable ideas and remarks and Professor V.A. Liskevich for invigorating discussions and constant attention to the work.

## References

[1] R. Bellman, Stability Theory of Differential Equations, McGraw–Hill, New York, 1953, xiii+166 pp.
[2] H. Brezis, L. Veron, Removable singularities for some nonlinear elliptic equations, Arch. Ration. Mech. Anal. 75 (1) (1980/81) 1–6.
[3] V.V. Chistyakov, On properties of solutions of semilinear second-order parabolic equations, J. Soviet Math. 60 (5) (1992) 1694–1724.
[4] D. Gilbarg, N.S. Trudinger, Elliptic Partial Differential Equations of Second Order, Springer-Verlag, Berlin, 2001, xiv+517 pp.
[5] J.B. Keller, On solutions of $\Delta u = f(u)$, Comm. Pure Appl. Math. 10 (1957) 503–510.
[6] I.T. Kiguradze, T.A. Chanturiya, Asymptotic Properties of Solutions of Nonautonomous Ordinary Differential Equations, translated from the 1985 Russian original, Math. Appl. (Soviet Series), vol. 89, Kluwer Academic, Dordrecht, 1993, xiv+331 pp.
[7] V.A. Kondrat'ev, E.M. Landis, Qualitative properties of the solutions of a second-order nonlinear equation, Mat. Sb. 135 (177) (1988) 346–360, 415 (in Russian); translation in Sb. Mat. 63 (2) (1989) 337–350.
[8] V. Kondratiev, V. Liskevich, Z. Sobol, Positive super-solutions to semi-linear second-order non-divergence type elliptic equations in exterior domains, Trans. Amer. Math. Soc., in press.
[9] V. Kondratiev, V. Liskevich, Z. Sobol, Positive solutions to semi-linear and quasi-linear second-order elliptic equations on unbounded domains, in: Handbook of Differential Equations: Stationary Partial Differential Equations, vol. 6, 2008, pp. 177–268.
[10] V.A. Kondrat'ev, V.A. Nikishkin, Isolated singularities of the solutions of equations of Emden–Fowler type, Differ. Uravn. 29 (6) (1993) 1025–1038, 1101 (in Russian); translation in Differ. Equ. 29 (6) (1993) 881–893.
[11] A. Kon'kov, Behavior of solutions of nonlinear second-order elliptic inequalities, Nonlinear Anal. 42 (7) (2000) 1253–1270.
[12] A. Kon'kov, Behavior of solutions of quasilinear elliptic inequalities, Sovrem. Mat. Fundam. Napravl. 7 (2004) 3–158 (in Russian).
[13] A. Kon'kov, On non-extendable solutions of ordinary differential equations, J. Math. Anal. Appl. 298 (1) (2004) 184–209.
[14] A. Kon'kov, Comparison theorems for second-order elliptic inequalities, Nonlinear Anal. 59 (4) (2004) 583–608.
[15] A. Kon'kov, Comparison theorems for elliptic inequalities with a non-linearity in the principal part, J. Math. Anal. Appl. 325 (2) (2007) 1013–1041.
[16] N. Meyers, J. Serrin, The exterior Dirichlet problem for second order elliptic partial differential equations, J. Math. Mech. 9 (1960) 513–538.
[17] R. Osserman, On the inequality $\Delta u \geqslant f(u)$, Pacific J. Math. 7 (1957) 1641–1647.
[18] M.H. Protter, H.F. Weinberger, Maximum Principles in Differential Equations, Springer-Verlag, New York, 1984, x+261 pp.
[19] S.D. Taliaferro, Asymptotic behavior of solutions of $y'' = \varphi(t)y^\lambda$, J. Math. Anal. Appl. 66 (1) (1978) 95–134.
[20] J.L. Vazquez, An a priori interior estimate for the solutions of a nonlinear problem representing weak diffusion, Nonlinear Anal. 5 (1) (1981) 95–103.
[21] J.L. Vazquez, L. Veron, Isolated singularities of some semilinear elliptic equations, J. Differential Equations 60 (3) (1985) 301–321.
[22] L. Veron, Singular solutions of some nonlinear elliptic equations, Nonlinear Anal. 5 (3) (1981) 225–242.
[23] L. Veron, Singularities of Solutions of Second Order Quasilinear Equations, Pitman Research Notes in Math. Ser., vol. 353, Longman, Harlow, 1996, viii+377 pp.